2D Transformation

Topics: *Affine Transformation Matrices, Homogeneous Coordinates*

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We begin by representing a point (x, y) in two-dimensional Cartesian coordinate system by a column vector i.e., a 2×1 matrix as $\begin{bmatrix} x \\ y \end{bmatrix}$.

We recall that a linear expression of the form $ax + by$ can be written as a matrix multiplication as $\begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$. This can easily be extended to a system of linear equations.

e.g., if we have the linear equations:

$$x' = a_1 x + b_1 y$$
$$y' = a_2 x + b_2 y$$

we can equivalently express this as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

Q1. What is 2D transformation?

Ans. The 2D geometric transformation means a change in either position or orientation or size or shape of some geometric object like line, polygon etc.

Q2. What is affine transformation?

Ans. A geometric transformation is called affine transformation if it satisfies the following:

- preserves collinearity i.e., all points lying on a line will still lie on a line after the transformation
- ratios of the distances is preserved i.e., ratio of the lengths of two line segments of a line remain same
- parallel lines remain parallel

But it may not preserve angles or distances. Translation, rotation, scaling, reflection, sheer are all affine transformations.

NB: Transformations involving only translation, rotation and scaling also preserves the angle between two lines and the length of a line segment.

Q3. Obtain the transformation matrices for the basic 2D affine transformations.

Ans. For simplicity, assume that the transfomations are applied with origin $(0, 0)$ as the reference point.

1. **Translation**
   This is also known as shift or displacement, here we specify two displacement values t_x and t_y along the two axes. In figure 1 we have a rectangular shape which is translated to its new position by $t_x = 2, t_y = 3$. Each point (x, y) of such an shape can be translated as per the following equations:

   $$x' = x + t_x$$
   $$y' = y + t_y$$

   alternatively we can write: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$

   NB: It is worth noting that unlike other transformations discussed below, translation is position independant.

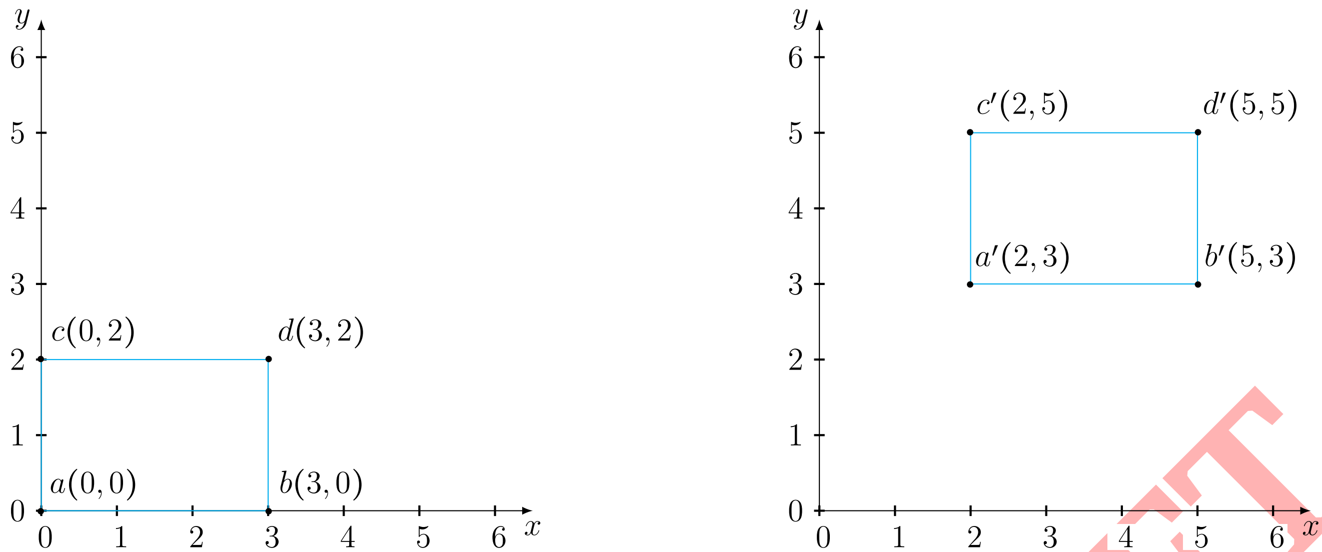

Figure 1: Before and after translation of the rectangle with $t_x = 2, t_y = 3$

2. **Rotation**

For rotation we need a fixed reference point about which the objects are rotated. For simplicity let us take origin $(0,0)$ as that reference point. We also choose counter clockwise (CCW) as default direction of the rotation. Recall that for any point p having Cartesian coordinates (x,y) and polar coordinates (r,θ) we have the following relations:

$x = r\cos\theta$ $r = \sqrt{x^2 + x^2}$ the distance from the origin

$y = r\sin\theta$ $\theta = \tan^{-1}\frac{y}{x}$ the slope or counter clockwise angle with +ve x-axis

Now, if we apply a rotation α (about the origin) the new point will have the polar coordinates $(r, \theta+\alpha)$ as shown in figure 2. Therefore we have,

$$x' = r\cos(\theta+\alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha = x\cos\alpha - y\sin\alpha$$
$$y' = r\sin(\theta+\alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = y\cos\alpha + x\sin\alpha$$

or, $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

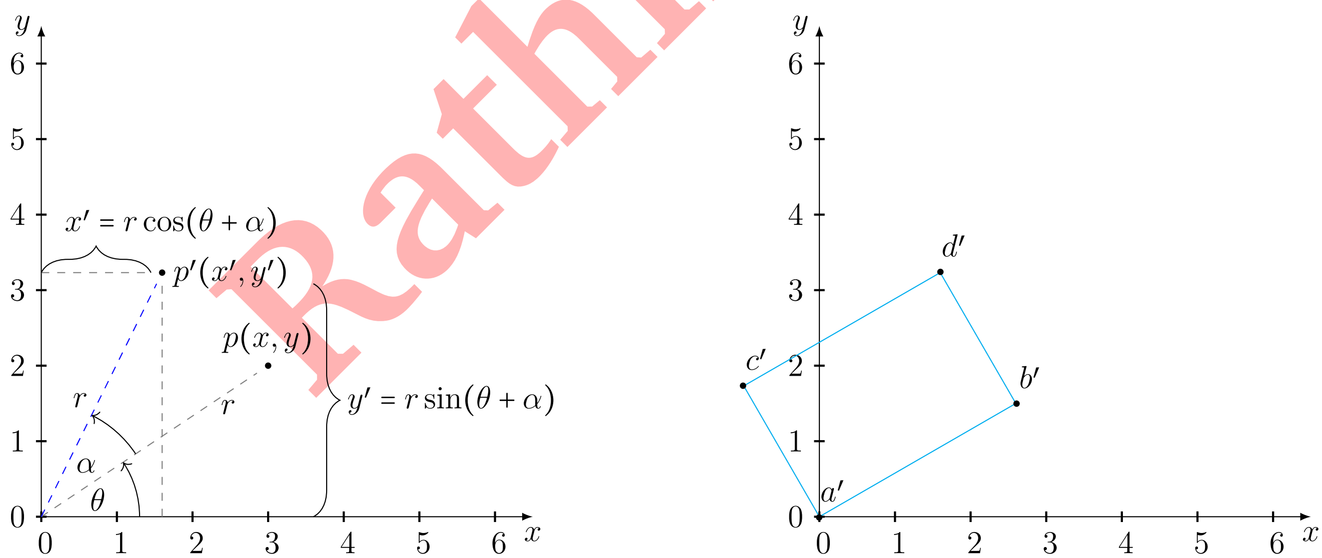

Figure 2: After rotation about the origin with $\alpha = 30°$

3. **Scale**
   A scale value larger than 1 enlarges a shape while scale value lesser than 1 shrinks it. We fix origin as the reference point as can be seen in figure 3. The scaling equations are given as follows:

$$x' = x\ s_x$$
$$y' = y\ s_y$$

this can be written as, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$

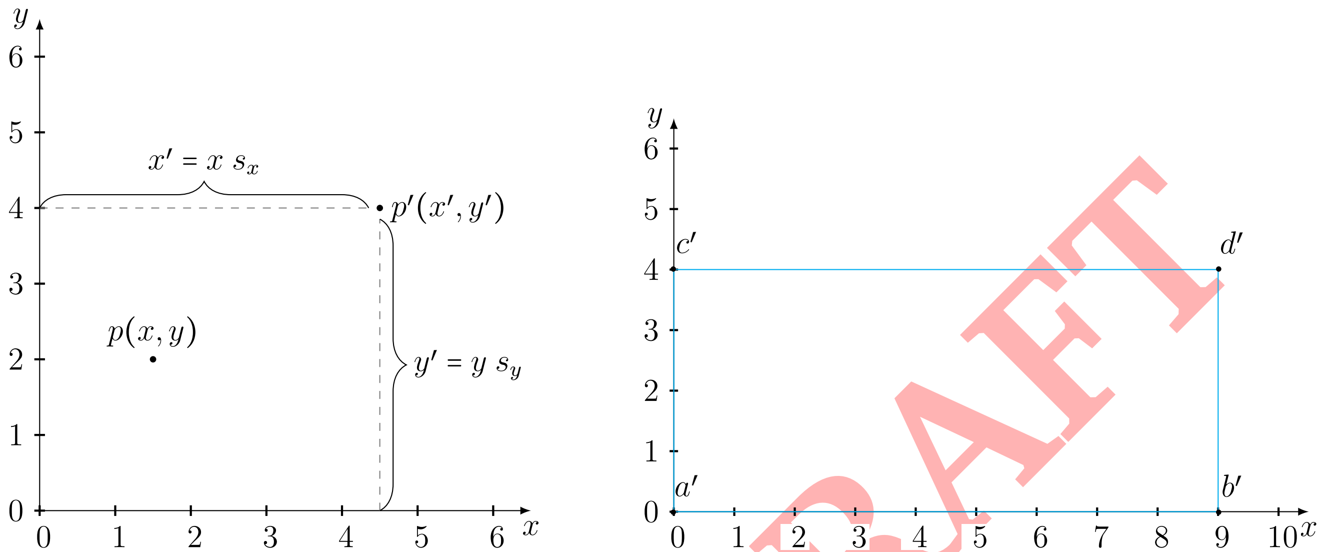

Figure 3: After scaling with $s_x = 3, s_y = 2$

4. **Shear**
   In shear transformation all points along a given line L remain fixed while other points are shifted parallel to L by a distance proportional to their perpendicular distance from L. It can be thought as an application of some force parallel to L which distorts the shape, where magnitude of the force increases as we move further away from the line L. Figure 4 depicts a sheer along x-axis and y-axis. The combined equations for shear along both axes is given as follows:

$$x' = x + y\ sh_x$$
$$y' = x\ sh_y + y$$

this can be written as, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$

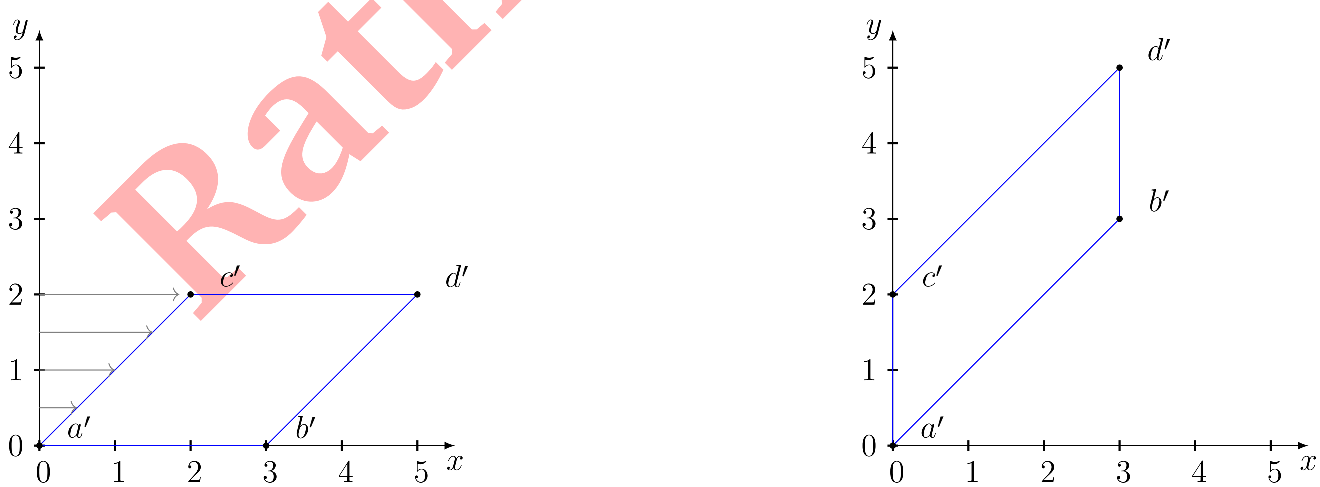

Figure 4: After shear with $sh_x = 1$ (left fig) and $sh_y = 1$ (right fig)

5. **Reflection**

Here we consider a line as a reference for the imaginary mirror which reflects an object.

(a) Reflection wrt x-axis: The line equation of x-axis is $y = 0$. The translation equations can be written as:
$x' = x$
$y' = -y$
this can be written as:
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) Reflection wrt y-axis: The line equation of y-axis is $x = 0$. The translation equations can be written as:
$x' = -x$
$y' = y$
this can be written as:
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) Reflection wrt origin: Here the considered mirror is a point (origin) instead of a line. The effect is analogous to the upside down image that we see in the physics experiment when light passes through a small pinhole. The translation equations can be written as:
$x' = -x$
$y' = -y$
this can be written as:
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

(d) Reflection wrt $y = x$: The translation equations can be written as:
$x' = y$
$y' = x$
this can be written as:
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

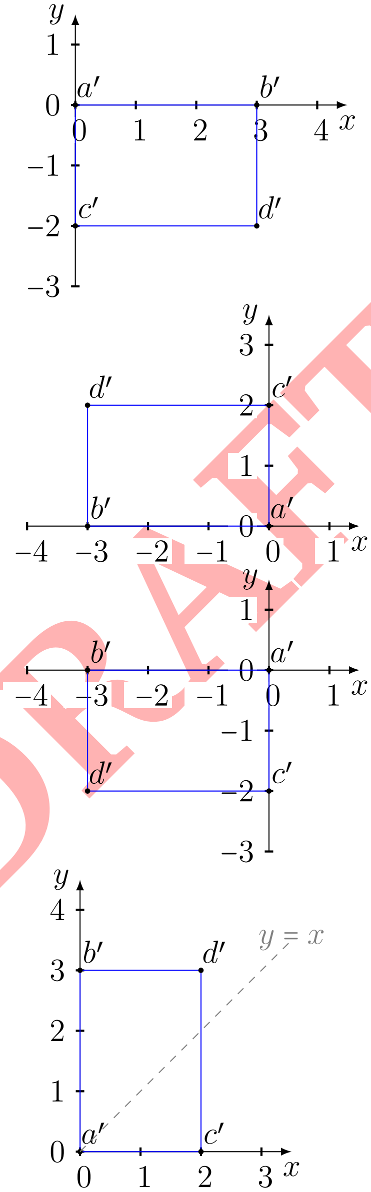

Figure 5: Reflection with respect to the x-axis ($y = 0$), y-axis ($x = 0$), origin and line $y = x$ respectively

NB: The first three cases is equivalent to a scaling with $s_y = -1$, $s_x = -1$ and $s_x = s_y = -1$ respectively.