Before proceding any further, let us observe that a system of linear equations involving two variables can be extended by adding an extra dummy variable d ~~whose coefficient is always 1~~ as follows:

$$\begin{aligned} f_1 &= a_1 \cdot x + b_1 \cdot y \\ f_2 &= a_2 \cdot x + b_2 \cdot y \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} f_1 &= a_1 \cdot x + b_1 \cdot y + 0 \cdot d \\ f_2 &= a_2 \cdot x + b_2 \cdot y + 0 \cdot d \\ d &= 0 \cdot x + 0 \cdot y + 1 \cdot d \end{aligned}$$

Thus we now have,
$$\begin{bmatrix} f_1 \\ f_2 \\ d \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ d \end{bmatrix}$$

Q. What is homogeneous coordinate systems?

→ A point (x, y) in 2D plane can be represented as (xh, yh, h) in homogeneous coordinate system where h can be any non-zero value.

Thus a point, say $(3,2)$, has infinitely many possible representation in homogeneous coordinate system e.g. $(3,2,1)$, $(6,4,2)$, $(30,20,10)$, $(1, 2/3, 1/3)$ etc.

This extra coordinate h is known as a weight which is homogeneously applied to each cartesian components namely x and y.

Such a coordinate can be written in matrix form as

$$\begin{bmatrix} xh \\ yh \\ h \end{bmatrix} \quad \text{or} \quad h \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For 2D transformation purposes we simply assume h to be 1.

Q. Why homogeneous coordinate is significant / what is its use?

→ Expressing positions in homogeneous coordinates allows us to represent all geometric transformations equations uniformly as matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = T \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

here T is the 3×3 transformation matrix.

Q. Write transformation matrices for basic affine transformation in homogeneous coordinate system.

① translation :

$$x' = x + t_x$$
$$y' = y + t_y$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \overset{T}{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

② rotation :

$$x' = x\cos\alpha - y\sin\alpha$$
$$y' = x\sin\alpha + y\cos\alpha$$

$$\Rightarrow T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ scale :

$$x' = x \cdot s_x$$
$$y' = y \cdot s_y$$

$$\Rightarrow T = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

④ shear :

$$x = x + y \cdot sh_x$$
$$y = y$$

$$\Rightarrow T = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(shear along x-axis)

$$x = x$$
$$y = y + x \cdot sh_y$$

$$\Rightarrow T = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(shear along y-axis)

⑤ reflection :

(x-axis)

$$x' = x$$
$$y' = -y$$

$$\Rightarrow T \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⋮

[complete yourself]

Q. : What is inverse transformation? Write inverse transformation matrices for the basic affine transformations.

→ For each geometric tranformation T there exists an inverse tranformation T^{-1} such that $TT^{-1} = T^{-1}T = I$ where I is the identity transformation matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⇒ for ~~scaling~~ translation, if $T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$ then $T^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$

i.e. inverse of translation is translation in opposite direction.

for rotation by angle α, the inverse should be a rotation of $(-\alpha)$

i.e. $T^{-1} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

for scaling we must have $T^{-1} = \begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$ // opposite double must be half //

for shear we have $T^{-1} = \begin{bmatrix} 1 & -sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for shear along x-axis)

and $T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (- - - - y-axis)

for relflection interestingly repeating the same reflection gives back the original.

Q. Verify in each of the above cases that $T \cdot T^{-1} = T^{-1} \cdot T = I$.

→ (homework)

Q. What is composite transformation?

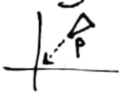
→ In many cases, we usually have to apply multiple basic transformation in succession to get the desired effect.

Let $P' = T_n(T_{n-1}(\cdots(T_2(T_1 \times P))))$ where T_i are transformation matrices applied in series, $1 \leq i \leq n$

Then $T_{comp} = T_n \times T_{n-1} \times \cdots \times T_2 \times T_1$ is the composite transformation matrix and $P' = T_{comp} \times P$.

Q. Derive the composite transformation matrix for rotation about any general pivot point.

→ Let ~~[illegible]~~ $P(x_p, y_p)$ be the pivot point about which we want to rotate some object. For this we follow the following steps:



1. we apply a translation $t_x = -x_p$, $t_y = -y_p$ to place the pivot point onto the origin. The transformation matrix is

$$T_1 = \begin{bmatrix} 1 & 0 & -x_p \\ 0 & 1 & -y_p \\ 0 & 0 & 1 \end{bmatrix}$$

2. Then we apply the required rotation (α) about the origin.

$$T_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. we back translate / apply inverse translation of T_1

$$T_3 = T_1^{-1} = \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

The composite transformation matrix is $T_{comp} = T_3 \times T_2 \times T_1$

[H/w: do the multiplication]

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & (1-\cos\alpha)x_p + \sin\alpha\, y_p \\ \sin\alpha & \cos\alpha & (1-\cos\alpha)y_p - \sin\alpha\, x_p \\ 0 & 0 & 1 \end{bmatrix}$$

Q. Derive the composite transformation matrix for scaling with general fixed point.

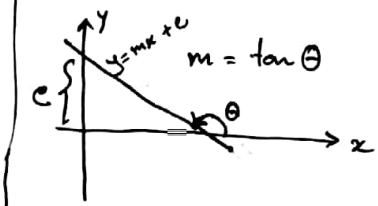
→ hint: the similar 3 steps: $T_{comp} = T_3 \times T_1 \times T_1$

[H/w: complete this]

$$= \begin{bmatrix} s_x & 0 & (1-s_x)x_p \\ 0 & s_y & (1-s_y)y_p \\ 0 & 0 & 1 \end{bmatrix}$$

Q. Derive the composite transformation matrix for reflection through an arbitrary line.

recall the pic



→ Let the line equation be $y = mx + c$

We follow the following steps:

1. translate the line so that it passes through the origin. $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$

2. rotate about origin so that the line coincide with x-axis

   i.e. $\alpha = -\theta$ $\therefore T_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $\theta = \tan^{-1} m$

3. reflect about x-axis $T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. ~~back~~ ~~rotate~~ rotate back by applying T_2^{-1} or rotation of $+\theta$

   $T_4 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. translate back $T_5 = T_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

Hence, $T_{comp} = T_5 \times T_4 \times T_3 \times T_2 \times T_1$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & -2c\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & c(\cos^2\theta - \sin^2\theta) + c \\ 0 & 0 & 1 \end{bmatrix}$$

(H/W: do the multiplication)

$$= \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m & -2mc \\ 2m & m^2-1 & 2c \\ 0 & 0 & 1 \end{bmatrix}$$

[see fig 4.29]

$\therefore 2\sin\theta\cos\theta$

$= \cos^2\theta \cdot 2\tan\theta$

$= \frac{2m}{1+m^2}$

$\therefore \cos^2\theta - \sin^2\theta$

$= \cos^2\theta(1 - \tan^2\theta)$

$= \frac{1-m^2}{1+m^2}$

recall,

$\frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta$

$= 1 + m^2$

$\therefore$ ~~$2\sin\theta\cos\theta = \frac{2\sin\theta\cos\theta}{\cos\theta}$~~

$\therefore \cos^2\theta = \frac{1}{1+m^2}$

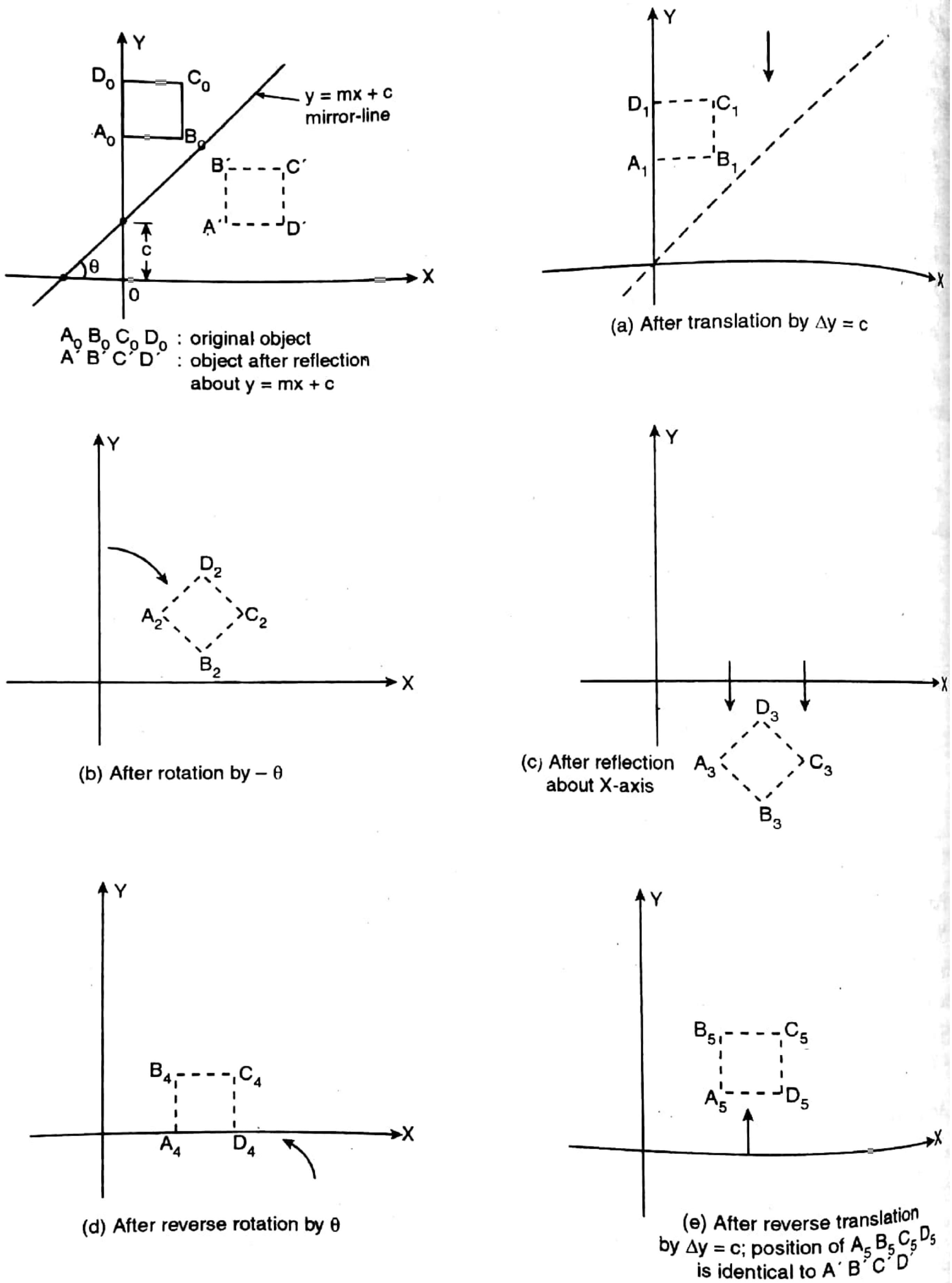
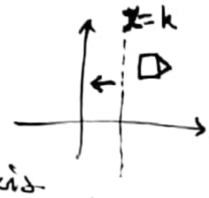


Fig. 4.29

src: Introduction To Computer Graphics And Multimedia, by Arup Chatterjee

The above procedure works only when the line $y = mx + c$ makes a finite intercept c by intersecting the y-axis. This is ~~very~~ not the case when the line is parallel to y-axis. In that case we have the line equation as $x = k$

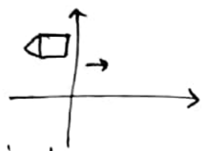
In this case we proceed as follows.



1. translate so that the line coincide with y-axis.

$$T_1 = \begin{bmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. reflect through y-axis $\qquad T_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



3. translate back $\quad T_3 = T_1^{-1} = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence here $T_{comp} = T_3 \times T_2 \times T_1$

Q. Consider a triangle with vertices at $(2,2)$, $(4,2)$, $(4,4)$. Rotate 90° about origin then reflect about the line $y = -x$

→ Here the rotation matrix is $T_1 = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and reflection matrix is $T_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The new vertices will be at:

$$T_2 \times T_1 \times \begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -4 \\ 2 & 2 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

(hw: do the multiplication)

New vertices are. $(-2,2)$, $(-4,2)$, $(-4,-4)$.

Q. Consider a polygon $A(20,10)$, $B(60,10)$, $C(60,30)$, $D(20,30)$.
Transform this shape by scaling it by 2 so that point A remains at the same place. How the area of the transformed shape changes?

→ [H/w: Do it yourself (DYI)]

{Ans: (20,10), (100,10), (100,50), (20,50) ; area is four times big}

Q. A triangle has its vertices located at $P(80,50)$, $Q(60,20)$, $R(100,10)$.

(a) Reflect this triangle through a line passing through $A(30,10)$ and parallel to y-axis

(b) Reflect this triangle PQR about its center (center of gravity) point.
(ভরকেন্দ্র)

→ [H/w: DYI]

{Ans: (a) (-20,50), (0,20), (-40,10)}

⇛ The center gravity (CoG) of this triangle is $\left(\frac{1}{3}(80+60+100),\ \frac{1}{3}(50+20+10)\right)$
[hint]
or $(80,30)$

1. translate by $(-80,-30)$ to place the (CoG) on the origin
2. reflect ~~about~~ through origin
3. translate back by $(80,30)$

[H/w: complete this]

Ans:

Q. Find the reflected view of the triangle having vertices at (30,40), (50,50), (40,70) through the mirror line passing through (20,0) and (0,20)

→ [H/w: DYI]

{Ans: (-20,-10), (-30,-30), (-50,-20)}

Q. Show that two succesive translations are commutative.

→ Let the two translation matrices be

$$T_1 = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 1 & 0 & t_x' \\ 0 & 1 & t_y' \\ 0 & 0 & 1 \end{bmatrix}$$

we need to show that $T_1 \times T_2 = T_2 \times T_1$.

$$LHS = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & t_x' \\ 0 & 1 & t_y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x + t_x' \\ 0 & 1 & t_y + t_y' \\ 0 & 0 & 1 \end{bmatrix}$$

$$RHS = \begin{bmatrix} 1 & 0 & t_x' \\ 0 & 1 & t_y' \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x' + t_x \\ 0 & 1 & t_y' + t_y \\ 0 & 0 & 1 \end{bmatrix}$$

So we have LHS = RHS (proved)

Q. Show that two succesive rotations are commutative. [H/W: DYI]

Q. - - - - - - - - scalings - - - - - - - [H/W: DYI]

Q. Show that rotation and scaling are commutative when either $s_x = s_y$ or $\alpha = n\pi$ for some integer n

→ Let the rotation and scaling transformation matrices be

$$T_R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we need to show that $T_R \times T_S = T_S \times T_R$

~~[do it step by step]~~

$$LHS = \begin{bmatrix} s_x \cos\alpha & -s_y \sin\alpha & 0 \\ s_x \sin\alpha & s_y \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ ~~RHS = (same matrix)~~

~~∴ LHS = RHS (proved)~~

$$RHS = \begin{bmatrix} s_x \cos\alpha & -s_x \sin\alpha & 0 \\ s_y \sin\alpha & s_y \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

case1: when $s_x = s_y$ (uniform scaling) we clearly have LHS = RHS

case2: when $\alpha = n\pi$ for some interger n we have $\sin\alpha = 0$ [and $\cos\alpha = \pm 1$]

{+1 when n is even, −1 when n is odd}

here
$$LHS = \begin{bmatrix} s_x \cos\alpha & 0 & 0 \\ 0 & s_y \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = RHS \text{ (proved)}$$

Q. Show that rotation and translation are not commutative in general.

→ It is sufficient to present one such case where these two transformations results into different values when applied in two different orders.

Let P be a point at (1,2) $P = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

and we consider a translation of (3, 4) $T_t = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

and rotation of 90° $T_R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We have,

$$T_R \times (T_t \times P) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 1 \end{bmatrix}$$

and

$$T_t \times (T_R \times P) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

clearly these two are two different points. (proved)

Q. Show that the three basic reflections can be achieved by only applying some suitable scaling.

→ The transformation matrix of reflection through x-axis (y=0 line) is :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

this is same as applying $s_x = 1$, $s_y = -1$

The - - - - - - - - - - - - - y-axis (x=0 line) is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

this is same as applying $s_x = -1$, $s_y = 1$

The - - - - - - - - - - - - - origin ((0,0) point) is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

this is same as applying $s_x = -1$, $s_y = -1$

Q. Show that reflection along the line $y=x$ is equivalent to the reflection along x-axis followed by a rotation of $90°$.

→ The transformation matrix for reflection along the line $y=x$ is:

$$T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The - - - - - - - for reflection along x-axis is:

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- - - - - - - - - for rotation of $90°$ is: $T_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(mind the order)

Now $T_3 \times T_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T_1$ (proved)

Q. Show that the reflection along the line $y=x$ is also equivalent to a rotation $90°$ followed by a reflection along y-axis.

→ [H/W : DYI]

Q. Show that the reflection along the line $y=-x$ is equivalent to ~~the~~ a reflection along x-axis followed by a rotation of $-90°$ (or $270°$)

and a rotation of $-90°$ followed by a reflection along y-axis.

→ [H/W : DYI]

Q. Show that reflection through the origin is equivalent to a rotation. Find the required angle of the rotation.

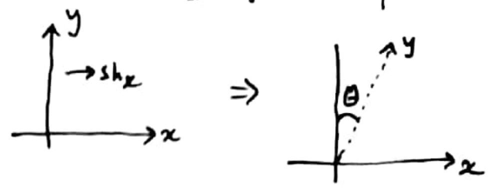
→ [H/W : DYI]

10

Q. Show that shear along x-axis is equivalent to following four steps.

1. rotate by $\theta/2$
2. scale with $s_x = \sin\theta/2$, $s_y = \cos\theta/2$
3. rotate by $-45°$
4. scale with $s_x = \sqrt{2}/\sin\theta$, $s_y = \sqrt{2}$

where θ is the shear angle



gradient of the sheared new y-axis would be $\tan(\pi/2 - \theta)$ or $\cot\theta$

→ [H/w : DYI]

Q. How would the transformation matrices and their operation change if we represent a point by a row vector instead of a column vector.

→ Let P be a point represented in colum vector as $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and T be some transformation matrix by which we obtain the transformed point matrix $P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$

We thus have. $P' = T \times P$

taking transpose on both sides of the equation we get,

$$(P')^T = (T \times P)^T = P^T \times T^T$$

or, $$[x' \;\; y' \;\; 1] = [x \;\; y \;\; 1] \times T^T$$

Thus, if we use row vector representation of the point then we simply transpose the transformation matrix (for colum vector form) and multiply it as the right hand operand (instead of left).

And for a series of transformations we have,

$$(P')^T = (T_n \times T_{n-1} \times \cdots \times T_2 \times T_1 \times P)^T = P^T \times T_1^T \times T_2^T \times \cdots \times T_{n-1}^T \times T_n^T$$